

Common physical mechanism for integer and fractional quantum Hall effects

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Abstract

Integer and fractional quantum Hall effects were studied with different physics models and explained by different physical mechanisms. In this paper, the common physical mechanism for integer and fractional quantum Hall effects is studied, where a new unified formulation of integer and fractional quantum Hall effect is presented. Firstly, we introduce a 2-dimensional ideal electron gas model in the presence of strong magnetic field with symmetry gauge, and the transverse electric field ε_2 is also introduced to balance Lorentz force. Secondly, the Pauli equation is solved where the wave function and energy levels is given explicitly. Thirdly, after the calculation of the degeneracy density for 2-dimensional ideal electron gas system, the Hall resistance of the system is obtained, where the quantum Hall number ν is introduced. It is found that the new defined ν , called filling factor in the literature, is related to radial quantum number n and angular quantum number $|m|$, the different n and $|m|$ correspond to different ν . This provides unification explanation for integer and fractional quantum Hall effects. It is predicated that more new cases exist of fractional quantum Hall effects without the concept of fractional charge.

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1 Introduction

In 1879, E. H. Hall discovered that when a conductor carrying an electric current perpendicular to an applied magnetic field develops a voltage gradient which is transverse to both the current and the magnetic field. This phenomenon is called Hall effect. About 100 years later, Klaus von Klitzing, in 1980, made the unexpected discovery that, under low temperature and strong magnetic field, the Hall conductivity was exactly quantized[1], in which the Hall conductivity σ takes on the quantized values, i.e. $\sigma = \nu \frac{e^2}{h}$ and ν takes integer values, we call it quantum Hall effect (QHE). For this finding, von Klitzing was awarded the 1985 Nobel Prize in Physics[2]. Very soon after that, under much more low temperature, the fractional quantum Hall effect(FQHE) was experimentally discovered in 1982 by Daniel Tsui and Horst Stormer[3],

in which ν , called filling factor, takes fractional values. Each particular value of the magnetic field corresponds to $\nu = \frac{p}{q}$, where p and q are integers with no common factors. Here q turns out to be an odd number with the exception of two ν 's $5/2$ and $7/2$. The principal series of such fractions are $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}$, etc., and $\frac{2}{3}, \frac{3}{5}, \frac{4}{7}$, etc.. All fractions have an odd denominator. The effect was explained by Laughlin in 1983, using a novel quantum liquid phase that accounts for the effects of interactions between electrons. Tsui, Stormer, and Laughlin were awarded the 1998 Nobel Prize in physics for their work.

The theoretical study of integer and fractional quantum Hall effects has lasted about thirty years and has great influence on physics[5]. For examples, the theory of Laughlin which introduced several novel concepts in correlated quantum fluids, inspired analogous effects in other subfields of physics; the quantum Hall effect was generalized to four dimensions [6] in order to study the interplay between quantum correlations and dimensionality in strongly correlated systems; two-dimensional electron systems were modeled by strings interacting with D-branes [7], where the fractionally-charged quasi-particles and composite fermions were described in the language of string theory; an interesting analogy between the quantum Hall effect and black hole has been reported, and in particular, the edge properties of a quantum Hall effect system have been used to model black hole physics from the point of view of an external observer [8]. Important developments of the quantum Hall effect have also taken place from the theoretical point of view[9]-[12].

However, in the most studies, integer and fractional quantum Hall effects were studied with different physics models and explained by different physical mechanisms. In this paper, an alternative unification description for integer and fractional quantum Hall effects is given, in which pure quantum mechanics theory is used and concepts of the fractionally-charged quasi-particles and composite fermions are not necessary.

2 The Hall density of electron gas model under strong magnetic field

In this section, we study the wave function and energy levels for electrons moving in an electromagnetic field, and give the expectation value of the electron moving size. After a new definition of quantum Hall number ν , we give the Hall density explicitly.

The Dirac equation for an electron moving in electromagnetic field is usually expressed as,

$$i\hbar\frac{\partial}{\partial t}\psi = [c\vec{\alpha} \cdot (\vec{p} + \frac{e}{c}\vec{A}) - e\phi + \mu c^2\beta]\psi. \quad (1)$$

Let's consider the situation of the electron moving in $x - y$ plane, and uniform magnetic field applied in z -direction. With the symmetric gauge, i.e. $\vec{A} = (-\frac{B}{2}y, \frac{B}{2}x, 0)$, and the scale potential of electric field is $\phi = -\varepsilon_1x - \varepsilon_2y$, in which ε_1 is the longitudinal external field and

ε_2 is the transverse electric field which can balance Lorentz force, then the Pauli equation for a single electron moving in an external electromagnetic field is simplified as

$$i\hbar \frac{\partial}{\partial t} \psi = (H^0 + H')\psi \quad (2)$$

where

$$\begin{aligned} H^0 &= \frac{1}{2\mu}(\hat{p}_x - \frac{eB}{2c}y)^2 + \frac{1}{2\mu}(\hat{p}_y + \frac{eB}{2c}x)^2 + \frac{1}{2\mu}(\hat{p}_z)^2 + \frac{e\hbar B}{2\mu c}\sigma_z, \\ H' &= e\varepsilon_1 x + e\varepsilon_2 y \end{aligned} \quad (3)$$

the last term of H^0 is the Stern-Gerlach term.

Treating H' as a perturbative term, then in cylinder coordinate description, we find, after tedious calculation, the perturbative energy levels and the stationary wave function are as follows.

$$E_{N\lambda} = \frac{1}{2\mu}p_z^2 + (N + \frac{\lambda+1}{2})\frac{e\hbar B}{2\mu c} - \frac{\mu c^2(\varepsilon_1^2 + \varepsilon_2^2)}{2B^2}, \quad (4)$$

$$\psi_{Nm\lambda}(\rho, \varphi, z, s, t) = \phi_{N,m}(\xi, \varphi)\chi_\lambda(s)e^{-ip_z z/\hbar}e^{-iEt/\hbar}, \quad (5)$$

where

$$\begin{aligned} \phi_{N,m}(\xi, \varphi) &= \phi_{N,m}^{(0)} - \frac{\alpha}{4}(\eta - i)\sqrt{N+1}\phi_{N+1,m+1}^{(0)}(\xi, \varphi) \\ &+ \frac{\alpha}{4}(\eta + i)\sqrt{N}\phi_{N-1,m-1}^{(0)}(\xi, \varphi), \\ N &= n + \frac{|m|+m}{2}, \quad n = 0, 1, 2, \dots; m = 0 \pm 1 \pm 2 \dots; \lambda = -1, 1, \end{aligned} \quad (6)$$

and

$$\phi_{N,m}^{(0)} = \left[\frac{(n+|m|)!}{a^2 \pi n!} \right]^{1/2} e^{im\varphi} e^{-\xi^2/2} \sum_{k=0}^n \frac{(-1)^k \binom{n}{k}}{(k+|m|)!} \xi^{2k+|m|}, \quad (7)$$

in which n is radial quantum number, m is angular momentum quantum number along z direction, λ is the spin quantum number, $a = (2\hbar c/eB)^{1/2}$, $\xi = \frac{\rho}{a}$, $\eta = \varepsilon_1/\varepsilon_2$, $\alpha = \frac{\mu e \varepsilon_2}{\hbar^2} \left(\frac{2\hbar c}{eB} \right)^{3/2}$ and $\binom{n}{k}$ is the binomial coefficient.

Now, let's calculate the expectation value of the electron moving size $\pi < \rho^2 >$. Obviously, the expectation value does not depend on the factors $\chi_\lambda(s)e^{-ip_z z/\hbar}e^{-iEt/\hbar}$ of the wave function, but depends on the function of $\phi_{N,m}(\xi, \varphi)$. Namely,

$$\begin{aligned} < S >_{Nm} = < \pi \rho^2 >_{Nm} = \pi a^2 \int_0^{2\pi} d\varphi \int_0^\infty |\phi(\xi, \varphi)|^2 \xi^3 d\xi \\ &= (2N+1-m)[1 + (2N+1)\beta] \frac{\hbar c}{eB} + \beta \frac{\hbar c}{eB}, \end{aligned} \quad (8)$$

where $\beta = \mu c^2(\varepsilon_1^2 + \varepsilon_2^2)/2B^2$. This shows that the expectation value of electron moving size depends on the state quantum number N, m and the external magnetic field.

The quantum Hall effect is the quantum effect when the magnetic field is very strong and the temperature is very low. In this condition the electrons will be fully polarized, and quantum number $\lambda = -1$, and the $m \leq 0$. Because $\beta \ll 1$, so when n and m are not very big, equation (8) can be written as

$$\begin{aligned} \langle S \rangle_{Nm} &= (2n + |m| + 1)[1 + (2n + m + |m| + 1)\beta] \frac{\hbar c}{eB} + \beta \frac{\hbar c}{eB} \\ &\approx (2n + |m| + 1) \frac{\hbar c}{eB}. \end{aligned} \quad (9)$$

When the energy level is chosen, the number of electron states is determined by quantum number m . In the region of $\langle S \rangle_{Nm}$, angular quantum number can be $0, -1, -2, \dots, -|m|$, i.e. $|m| + 1$ values, so the energy degeneracy density (energy degeneracy on per unit area) for electron gas in strong magnetic field is

$$n_B = \frac{|m| + 1}{\langle S \rangle_{Nm}} = \frac{|m| + 1}{2n + |m| + 1} \frac{eB}{\hbar c}, \quad (10)$$

in which n_B is also called Hall density.

Now let's define a quantum Hall number as

$$\nu = \frac{|m| + 1}{2n + |m| + 1}, \quad (11)$$

called filling factor in the literature, then the Hall density has the form of

$$n_B = \nu \frac{eB}{\hbar c}. \quad (12)$$

This shows that Hall density is proportional to the strength of magnetic field.

3 Physical understanding of ideal electron gas model under the strong magnetic field

The conditions for quantum Hall effect are low temperature and strong magnetic field. When an electron moves in a magnetic field with the field strength $B = 200000$ Gauss, the interaction energy between electron and magnetic field is $\varepsilon_0 = e\hbar B/\mu c = 0.00232\text{eV}$. Though the energy is very small, it is much bigger than the dynamic energy of electron with velocity $v = 3 \times 10^5 \text{cm/s}$, $p_z^2/(2\mu) = 2.56 \times 10^{-5}\text{eV}$. So, in the presence of strong magnetic field, the electron moving size, in ground state, is 150 times than that of the electron in the ground state of hydrogen atom. Therefore, in strong magnetic field, the electrons have much freedom than they are in atoms, but the interaction between magnetic field and electron is much smaller than the interaction of electron with nuclear of atom. Thus, when the Lorentz force is balanced by a transverse

electric field ε_2 , the dilute electron gas in magnetic field can be considered as ideal electron gas system.

On the other hand, the interaction among electrons is materialized from the electric field ε_2 in y direction, which is produced by the nonuniform distribution of electrons along y direction. According to (9), the expectation value of the electron moving size proportional to $1/B$, that is, bigger B to corresponding less expectation value of the electron moving size. For the fractional quantum Hall effect, B is much stronger, and then expectation value of the electron moving size get extreme less, and less interaction among electrons. So in the situation of integer quantum Hall effect and fractional quantum Hall effect, the ideal electron gas model is reasonable.

After some calculations, we get the expectation value of electric currents as

$$\begin{aligned} J_x &= \langle j_x \rangle = -en_s \langle v_x \rangle = en_s \frac{c\varepsilon_2}{B}, \\ J_y &= \langle j_y \rangle = -en_s \langle v_y \rangle = -en_s \frac{c\varepsilon_1}{B}. \end{aligned} \quad (13)$$

From the equations above, we can understand why there exists superconductivity in x direction on Hall plateau.

Since we can define

$$\frac{n_s}{n_B} = i, \quad (14)$$

and

$$w = i\nu, \quad (15)$$

from the definition of Hall resistance and equation (10), we get the Hall resistance as

$$\rho_{xy} = -\rho_{yx} = \frac{B}{n_s e c} = \frac{1}{i\nu} \frac{h}{e^2} = \frac{1}{w} \frac{h}{e^2}. \quad (16)$$

Obviously, the Hall resistance only depends on the quantum Hall number, and the later depends on the ratio of $|m| + 1$ and $2n + |m| + 1$. This is the key result of this paper. With this result, the quantum Hall effect and fractional quantum Hall effect can be unified formulated.

4 Unification formulation of integer and fractional quantum Hall effects

In this section, we will use the results above to provide a unification description for both integer and fractional quantum Hall effects. From the formula of quantum Hall number (11), we can see that different quantum number n and m correspond to different quantum states, and that ν can take value 1 and also fraction, they relate respectively to integer quantum Hall effect and fractional quantum Hall effect.

Table 1: quantum Hall numbers when $n = 0, 1, \dots, 9$ and $m = 0, -1, \dots, -9$

$m \backslash n$	0	1	2	3	4	5	6	7	8	9
0	1	1/3	1/5	1/7	1/9	1/11	1/13	1/15	1/17	1/19
-1	1	1/2	1/3	1/4	1/5	1/6	1/7	1/8	1/9	1/10
-2	1	3/5	3/7	1/3	3/11	3/13	1/5	3/17	3/19	1/7
-3	1	2/3	1/2	2/5	1/3	2/7	1/4	2/9	1/5	2/11
-4	1	5/7	5/9	5/11	5/13	1/3	5/17	5/19	5/21	5/23
-5	1	3/4	3/5	1/2	3/7	3/8	1/3	3/10	3/11	1/4
-6	1	7/9	7/11	7/13	7/15	7/17	7/19	1/3	7/23	7/25
-7	1	4/5	2/3	4/7	1/2	4/9	2/5	4/11	1/3	4/13
-8	1	9/11	9/13	3/5	9/17	9/19	3/7	9/23	9/25	1/3
-9	1	5/6	5/7	5/8	5/9	1/2	5/11	5/12	5/13	5/14

When $n = 0$, from equation (11) we get $\nu = 1$. And now the angular quantum number can still take one of $|m| + 1$ values: $0, -1, -2 \dots -|m|$. In other words, on one energy level, the electrons of the electron gas system can fill different state with different angular quantum number $0, -1, -2 \dots -|m|$. The less absolute value of angular quantum number correspond to the more stable state and the less moving range. So, all electrons firstly occupy the state of angular quantum number equivalent to 0, and when the ground state is full filled, Hall plateau appears. When the magnetic field becomes less, electron density of electron gas also gets less, then the extra electrons of state $m = 0$ will fill the state of $m = -1$. When these states are full filled, the Hall plateau appears again. When the external magnetic field gets less and less, the electrons will fill the states of $m = -2, -3 \dots$, other Hall plateaus appear one by one. This is the integer Hall effect.

When $n \neq 0$, the electrons of the electron gas system stay in excited states. Different n and different $|m|$ correspond to different fractional quantum Hall number and thus correspond to different Hall effects, which is just the fractional Hall effect. Therefore, the fractional quantum Hall effect corresponds to the filling of electrons to excited states. The principle of the filling is the same as that of the integer quantum Hall effect filling the electrons to ground states. Hall plateaus also appear when the corresponding state of the chosen angular quantum number is full filled by electrons. Now, Table I shows possible values of Hall quantum numbers $\nu = \frac{|m|+1}{2n+|m|+1}$, where $n = 0, 1, 2 \dots, 9$ and $m = 0, -1, -2 \dots, -9$.

According to Table I and the analysis above, some remarks are made as follows.

1). The ν of all known fractional quantum Hall effects are concluded in the table. For examples, $\nu = \frac{1}{3}, \frac{2}{3}, \frac{2}{5}, \frac{3}{5}, \frac{1}{7}, \dots$, of which $\nu = \frac{1}{3}$ appears firstly in the table and also most frequently. From this point of view, its first discovering by experiment is easy understood, and also it corresponds to the excited state with the lowest energy.

2). $\nu = \frac{1}{2}$ also appears frequently in Table I. Its first corresponding quantum number is

$n = 1, m = -1$; the second is $n = 2, m = -3$; its other corresponding quantum number is $n = |m| - 1$. While, no Hall plateaus appear corresponding to $\nu = \frac{1}{2}$, which needs further studying.

3). When $m = 0$, the Hall quantum number is reduced to $\nu = 1/(2k + 1)$, and the electron gas system in these states are so called incompressible quantum liquid. In this case, electrons are in s-wave states, namely, in circular Bohr orbit.

4). Besides the quantum Hall numbers defined in (11) which fully described the known fractional Hall effect, the fractional Hall effect for other quantum Hall numbers may also exist. We expect they can be experimentally checked very soon.

5). From equation (15) we know when $n_s/n_B = i > 1$, the quantum Hall numbers can be obtained for improper fraction. For example, $n_s/n_B = i = 5, \nu = 1/2$ is the fractional quantum Hall effect for $\nu = 5/2$.

5 Conclusion remarks

The electron gas model is used in this paper to unify the integer and fractional quantum Hall effect, in brief, with the solution of the Schrodinger equation, we firstly got the wave function, moving size, and the expectation value of current density for the single-electron in strong magnetic field and weak electric field. Explicit calculations show that B , external magnetic field, is in proportion to Hall density n_B , but in inverse proportion to the expectation value of electron moving size $\langle S \rangle_{Nm}$, all electrons have almost the same possibility to meet with each other in both the conditions where the integer quantum Hall effect and the fractional quantum effect appear. This is the physical foundation of this paper where we use single electron model to describe the fractional quantum Hall effects. With $\nu = \frac{|m|+1}{2n+|m|+1}$, a unified interpretation is given to the integer and fractional quantum effects. This paper shows both effects have common physical mechanisms as follows. (1) Every electrons in dilute two-dimensional electron gas system is same as one single electron. (2) Energy levels for electrons with angular quantum number $m \leq 0$ is not related to m , so the energy level degeneracy is infinite theoretically, but in limited range, the degeneracy is finite. (3) Concepts of fractionally-charged quasi-particles and composite fermions are not used in this study. The electron gas model need to be further studied, for example, the energy levels and wave function for the electron gas system etc. will be reported in our forthcoming papers.

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